Coarse to Over-Fine Optical Flow Estimation

Tomer Amiaz* Eyal Lubetzky† Nahum Kiryati*

*School of Electrical Engineering
†School of Computer Science
Tel Aviv University
Tel Aviv 69978, Israel

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Abstract

We present a readily applicable way to go beyond the accuracy limits of current optical flow estimators. Modern optical flow algorithms employ the coarse to fine approach. We suggest to upgrade this class of algorithms, by adding *over-fine* interpolated levels to the pyramid. Theoretical analysis of the coarse to *over-fine* approach explains its advantages in handling flow-field discontinuities and simulations show its benefit for sub-pixel motion. By applying the suggested technique to various multiscale optical flow algorithms, we reduced the estimation error by 10%-30% on the common test sequences. Using the coarse to *over-fine* technique, we obtain optical flow estimation results that are currently the best for benchmark sequences.

Key words: optical flow

1 Introduction

Optical flow estimation is one of the central problems of computer vision. Optical flow is the apparent motion between two frames in a sequence (Horn, 1986). It is relevant in applications such as surveillance and tracking, and is part of higher level computer vision tasks such as shape from motion. Since the pioneering work of Horn and Schunck (1981) and Lucas and Kanade (1981), significant improvement has been achieved in the estimation of optical flow, with the best published results provided by Brox et al. (2004).

The estimation of optical flow relies on the assumption that objects in an image sequence may change position but their appearance remains the same (or nearly the same). Classically this is represented by the grey-level constancy assumption or the optical flow constraint. However, this assumption by itself is not sufficient for optical flow estimation. Horn and Schunck (1981) add a smoothness assumption to regularize the flow, and Lucas and Kanade (1981) assume constant motion in small windows.

Higher accuracy can be achieved using coarse-to-fine or multi-scale methods (Anandan, 1989; Black and Anandan, 1996; Mémin and Pérez, 2002; Brox et al., 2004). These methods accelerate convergence by allowing global motion features to be detected immediately, but also improve the accuracy of flow estimation because they provide a better approximation of image gradients via warping (Bruhn et al., 2005).

In this paper we suggest extending the multi-scale approach beyond the fine grid to *over-fine* levels. The representation of the image data on the over-fine grids is obtained by interpolation. Nevertheless, we show that computing optical flow on the extended pyramid improves optical flow accuracy. As described in the theoretical section of this paper, the *coarse to over-fine* approach leads to sharpening of the flow edges. Sub-pixel motion is observed to be more accurately estimated as well.

The suggested approach appears to be applicable to any multi-scale optical flow estimation algorithm. Its application to several algorithms was tested using standard benchmark synthetic sequences and a real-world sequence. Although it is relatively simple to upgrade an existing multi-scale algorithm using our approach, accuracy is substantially improved. The standard accuracy metric for optical flow algorithms is the average angular error (AAE) (Barron et al., 1994). Using the proposed *coarse to overfine* approach, the AAE is reduced by 10%-30% for conceptually different algorithms such as Brox et al. (2004) and Anandan (1989). In particular, applied to the algorithm of Brox et al. and its variants, the suggested approach yields results which are currently the best for several common test sequences.

The rest of this paper is organized as follows: Section 2 describes the details of the *coarse to over-fine* approach, and its theoretical foundation. We describe the assumptions on which most models for optical flow estimation algorithms are based, and summarize the concepts of two well known multi-scale algorithms by Anandan (1989) and Brox et al. (2004). Next, we describe our approach and analyze its advantages. Section 3 presents experimental results on the incorporation of the coarse to over-fine approach within the algorithms of Brox et al. (2004), Anandan (1989) and Amiaz and Kiryati (2006). Section 4 consists of a short discussion and concluding remarks.

2 Coarse to over-fine optical flow

2.1 Multi-scale optical flow estimation

We begin this section with a short description of several common assumptions used in optical flow estimation. We demonstrate their use in two well known algorithms, Brox et al. (2004) and Anandan (1989). While both algorithms incorporate a multi-scale approach, the first is a variational differential method (which gives the best published results on common test sequences), whereas the latter attempts to match regions in the image sequence.

Most algorithms for optical flow estimation are based on minimization of an objective functional $E(u, v)$, that includes two terms, a data term and a smoothness term. The data term, $E_d(u, v)$, measures the quality of the correspondence between the two frames, according to the given optical flow field. This measure relies on the *grey level constancy* assumption, which states that the grey level of objects is constant in consecutive frames. Let $I(x, y, t)$ denote the grey level of the (x, y) pixel in the t-th frame, and let $\vec{w} = (u, v, 1)$ denote the displacement (optical flow) vector of one frame, where u and v depend on x, y and t. Using these notations, this assumption takes the form:

$$I(x, y, t) = I(x + u, y + v, t + 1) \tag{1}$$

Note that first order Taylor expansion leads to the well known optical flow constraint:

$$I_x u + I_y v + I_t = 0 \tag{2}$$

The grey value constancy assumption is sensitive to noise in the images. It is customary to accommodate for this by pre-blurring the image or equivalently by using weighted windows around each pixel.

Finding the flow field by minimizing the data term alone is an ill-posed problem, since the optimum might be attained by many dissimilar displacement fields. In order to solve the problem, regularization is required. The most suitable regularization assumption is *piece-wise smoothness*, that arises in the common case of a scene that consists of semi-rigid objects. Mathematical representation of piecewise smoothness regularization is not trivial, and various approximations have been applied over the years, see Black and Anandan (1996); Brox et al. (2004); Amiaz and Kiryati (2006).

The multi-scale coarse-to-fine approach is used by most modern algorithms for optical flow estimation, in order to support large motion and for improved accuracy. This approach relies on estimating the flow in an image pyramid, where the apex is the original image at a coarse scale, and the levels beneath it are warped representations of the images based on the flow estimated at the preceding scale. This ensures that the small motion assumption of

Eq. (2) remains valid.

2.1.1 The differential method of Brox et al. (2004)

This is a variational method, based on minimizing the objective functional

$$E(u, v) = E_d(u, v) + \alpha E_s(u, v) \ , \tag{3}$$

where α balances the data term E_d with the smoothness term E_s.

The data term incorporates the grey value constancy assumption, as well as the *gradient constancy* assumption, according to which the gradient of the grey level of objects is constant in consecutive frames. This assumption accommodates for slight changes in the illumination of the scene. The definition of the data term is:

$$E_d(u, v) = \int L\left(|I(\vec{x} + \vec{w}) - I(\vec{x})|^2 + \gamma|\nabla I(\vec{x} + \vec{w}) - \nabla I(\vec{x})|^2\right) \mathrm{d}\vec{x} \ , \tag{4}$$

where $\nabla = (\partial x, \partial y)$ denotes the spatial gradient, γ relates the weight of the two constancy assumptions, and $L(s^2) = \sqrt{s^2 + \varepsilon^2}$ is a smooth approximation of the L_1 norm, $L_1(s) = |s|$. Using the L_1 norm rather than the common L_2 norm, the influence of outliers is reduced and estimation is robust, see e.g. Brox et al. (2004); Bar et al. (2006). The incorporation of the constant ε makes the approximation differentiable at $s = 0$; the value of ε sets the level of approximation. Notice that the data term remains convex, but the derivatives are highly non-linear, complicating the minimization process.

The smoothness term $E_s(u, v)$ is defined as:

$$E_s(u, v) = \int L\left(|\nabla u|^2 + |\nabla v|^2\right) \mathrm{d}\vec{x} \ . \tag{5}$$

The minimization of $E(u, v)$ is a nested iterative process, with external and internal iterations. The external iterations are with respect to scale. The internal iterations are used to linearize the Euler-Lagrange equations and solve the resulting linear set of equations. Linearization via fixed-point iterations is used both in the external and internal loops. The linear equations are solved using successive over relaxation (SOR) (Press et al., 1992).

An extension of this method was recently proposed, which uses the level set method to reduce the estimation error near flow discontinuities (Amiaz and Kiryati, 2006). This extension will be referred to in this paper as the Piecewise Smooth Flow algorithm.

2.1.2 The Anandan region matching method (Anandan, 1989)

The Anandan method is based on matching square regions of the image with neighboring square regions in the successive image, performed on a multi-scale pyramid. The distance measure used is SSD (sum of square differences). Keeping the above definitions for $I, \vec{w}, \vec{x}$, the measure is:

$$SSD(\vec{x}, \vec{w}) = \sum_{-n \le i,j \le n} W(i, j) \left(I\left(\vec{x} + (i, j, 0)\right) - I\left(\vec{x} + \vec{w} + (i, j, 0)\right)\right)^2 \ , \tag{6}$$

where W is a two-dimensional window function, and $(i, j, 0)$ is the vector used to scan the square region. The method's smoothness term is based on Horn and Schunck's term, with an additional limit on the difference between the displacement vector solutions in the current and previous levels of the pyramid.

2.2 Upgrading multi-scale optical flow algorithms

Consider a multi-scale optical flow estimation algorithm $\mathcal{A}$, whose objective functional $E(\mathcal{A})$ is comprised of a data term, E_d, and a smoothness term, E_s. The following modified algorithm $\mathcal{A}'$ yields a significantly more accurate flow than $\mathcal{A}$:

1. Let $I_0(x, y, t)$ denote the input sequence, and let $k = 1$ denote the iteration counter.

2. Execute $\mathcal{A}$ on the original image sequence I_0.

3. For each of the input frames, extend the image pyramid beyond the finest scale, using interpolation (resulting in a coarse-to-*over*-fine pyramid). The extension of the pyramid is an additional frame sequence, I_k, derived from the original sequence, I_{k-1}, in the following manner:

$$I_k(x, y, t) = I_{k-1}(x/2, y/2, t) \ . \tag{7}$$

   For non-integer indices $x/2, y/2$ of I_{k-1}, either bilinear or bicubic interpolation is used. The pre-blurring kernel used in algorithm $\mathcal{A}$ should be widened.

4. Continue the execution of $\mathcal{A}$ on the extended pyramid.

5. Sample the resulting flow back to the original grid:

$$\vec{w}(x, y) = \vec{w}^k(2^k x, 2^k y)/2^k \ . \tag{8}$$

   This also provides the value of the objective functional in the original scale, $E^k(\mathcal{A}')$.

6. Increment k and return to step 3, as long as $E^{k-1}(\mathcal{A}')$ is significantly larger than $E^k(\mathcal{A}')$, or subject to a predefined number of iterations.

2.3 Analysis of the coarse to over-fine approach

We claim that extending the image pyramid results in a lower penalty for discontinuities in the optical-flow field. Mathematical analysis of optical flow methods appears to be difficult. The focus of this paper is on the *over-fine approach*, as applied to optical flow estimation, rather than on optical flow per-se. We therefore consider an analogous curve fitting problem, and study the effect of the over fine approach on edge preservation. The problem we choose is estimating a smooth function based on samples of a step function. We show that a sharper edge is estimated on the over-fine grid.

We begin with the one-dimensional case, and consider the following minimization process: given equidistant samples $\vec{y} = (y_1, \ldots, y_{2n})$, we are interested in obtaining a set of points $\vec{z} = (z_1, \ldots, z_{2n})$ which minimizes

$$F(\vec{z}) = \sum_{i=1}^{2n}(z_i - y_i)^2 + \alpha \sum_{i=1}^{2n-1} (z_{i+1} - z_i)^2 \ . \tag{9}$$

where $\alpha > 0$ is the smoothing parameter.

To demonstrate the edge sharpening effect of the over-fine phase in smooth approximation, we consider the following scenario. Let the input be a step function:

$$y_i = \begin{cases} 1 & 1 \leq i \leq n \\ -1 & n+1 \leq i \leq 2n \end{cases} \tag{10}$$

Let $\vec{z}^*$ denote the minimizer of $F(\vec{z})$ according to equation (9). Clearly, since $F(\vec{z})$ is a non-negative sum of squares, it has a single minimum, attained at the point satisfying $\nabla F(\vec{z}) = 0$.

We shall denote the over-fine version of the problem with $\vec{y}_{of}, \vec{z}_{of}$, both of length $4n - 1$.

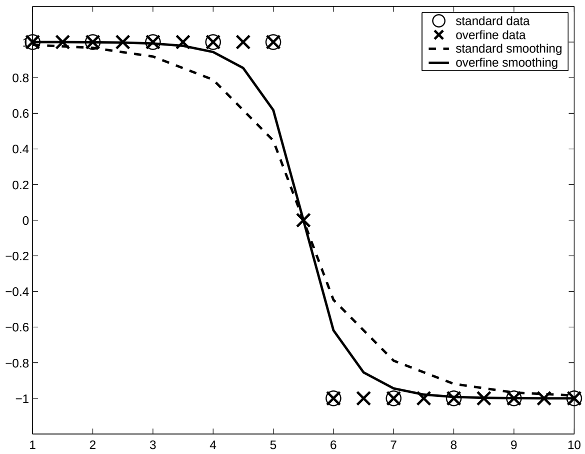


Figure 1: Smooth approximation of a sampled step-function. The over-fine approximation leads to a sharper edge than that of the original approximation version.

The interpolation of y_i results in

$$y_{of,i} = \begin{cases} 1 & 1 \leq i \leq 2n-1 \\ 0 & i = 2n \\ -1 & 2n+1 \leq i \leq 4n-1 \end{cases} , \tag{11}$$

and $\vec{z}^*_{of}$ is the minimizer of (9) with the summations extended to $4n-1$ and $\vec{y}_{of}$ replacing $\vec{y}$.

Figure 1 presents the original and over-fine versions of the data and the two smooth approximations. From the inherent antisymmetry of the problem we can deduce that the approximations $\vec{z}^*$ and $\vec{z}^*_{of}$ are antisymmetrical around the points $n+1/2$ and $2n$ respectively.

$$\begin{aligned} z^*_{n+1-i} &= -z^*_{n+i} \neq 0 \ , (i = 1, \dots, n). \\ z^*_{of,2n} &= 0 \\ z^*_{of,2n-i} &= -z^*_{of,2n+i} \neq 0 \ , (i = 1, \dots, 2n-1). \end{aligned} \tag{12}$$

We claim that the edge in the solution for $\vec{z}^*_{of}$ is sharper compared to $\vec{z}^*$, as the following equation reflects:

$$z^*_{of,2i-1} > z^*_i \ , (i = 1, \dots, n) \ . \tag{13}$$

The proof will proceed along the following lines: Using the antisymmetry of the problem, we will reduce it to a minimization problem on the left half of the domain and prove that the point nearest to the edge is higher in the over-fine version. Next, we shall prove by induction that all the values estimated by the over-fine version are higher than those estimated by the non over-fine version. Finally using the monotonicity of $z^*_{of,i}$ we show that the subsampling provides the desired result (13).

Inserting the antisymmetry property (12) into the minimization problems of $\vec{z}$ and $\vec{z}_{of}$, we obtain

$$\vec{z}^* = \underset{\vec{z}}{\operatorname{argmin}} \left[\sum_{i=1}^{n} (z_i - 1)^2 + \alpha \sum_{i=1}^{n-1} (z_{i+1} - z_i)^2 + 2\alpha \left(z_n\right)^2 \right] \tag{14}$$

$$\vec{z}^*_{of} = \underset{\vec{z}}{\operatorname{argmin}} \left[\sum_{i=1}^{2n-1} (z_i - 1)^2 + \alpha \sum_{i=1}^{2n-2} (z_{i+1} - z_i)^2 + \alpha \left(z_{2n-1}\right)^2 \right] \tag{15}$$

This immediately implies the desired result $z^*_{of,2n-1} > z^*_n$ due to the larger coefficient of the square of the latter in the minimized term (see Appendix for details).

Suppose now that the values of $z^*_{of,2n-1}, z^*_n$ have been determined. By inserting them into (14) and (15), we can obtain the relation between $z^*_{of,2n-2}$ and z^*_{n-1},

$$\vec{z}^* = \underset{\vec{z}}{\operatorname{argmin}} \left[\sum_{i=1}^{n-1} (z_i - 1)^2 + \alpha \sum_{i=1}^{n-2} (z_{i+1} - z_i)^2 + \alpha \left(z_{n-1} - z^*_n\right)^2 \right] \tag{16}$$

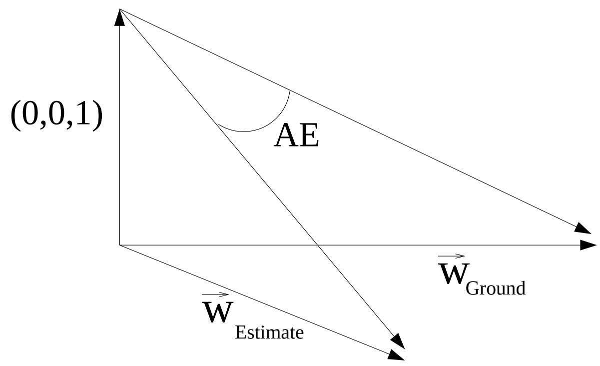


Figure 2: Angular error is the angle between the estimated flow and the true flow, with a unit perpendicular vector added to both.

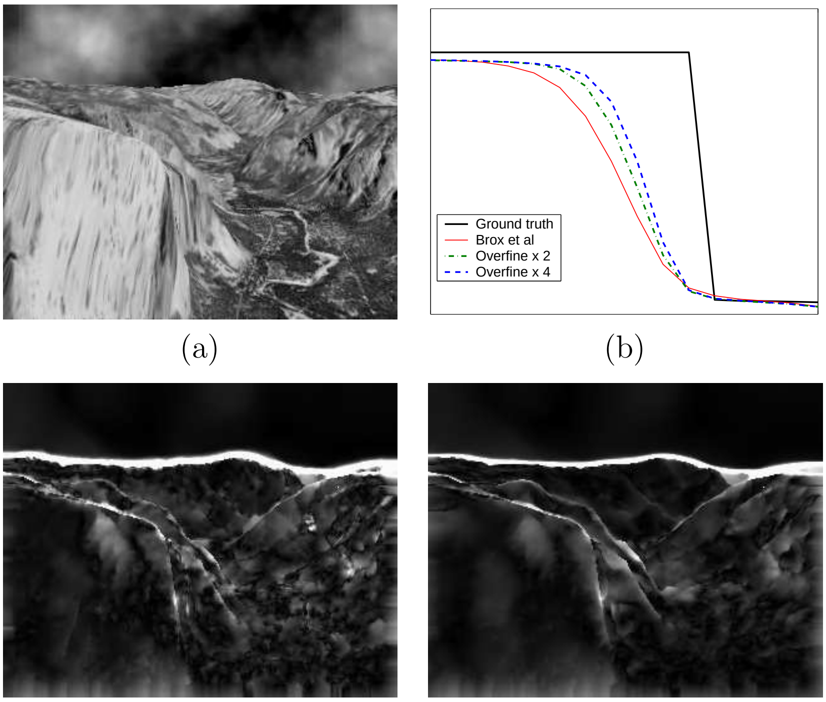
$$\vec{z}^*_{of} = \underset{\vec{z}}{\mathrm{argmin}} \left[\sum_{i=1}^{2n-2} (z_i - 1)^2 + \alpha \sum_{i=1}^{2n-3} (z_{i+1} - z_i)^2 + \alpha \left(z_{2n-2} - z^*_{of,2n-1} \right)^2 \right] \tag{17}$$

As can be seen, z^*_{n-1} is coupled to a smaller value than $z^*_{of,2n-2}$, and therefore (everything else being equal) it is smaller (see Appendix for rigorous proof). Repeating this process inductively results in $z^*_{of,2n-i} > z^*_{n+1-i}$ for $i = 1, \dots, n$. The values of $z^*_{of,i}$ are monotonically decreasing and thus we obtain $z^*_{of,2i-1} \geq z^*_{of,n+i-1} > z^*_i$ for $i = 1, \dots, n$, which is the desired result.

A similar argument to the one above extends the discontinuity scenario above to a 2-dimensional setting, demonstrating how the over-fine phase may provide sharper edges.

3 Experimental results

We present the quantifiable effect of the method on the well-known *Yosemite* and *Street with Car* (Galvin et al., 1998) sequences and the real-world *Flower Garden* sequence. Angular error was measured according to the method of Barron et al. (1994) (see figure 2). We demonstrate the effectiveness of the coarse to over-fine approach when applied to the algorithms of Anandan (1989), Brox et al. (2004), and the Piecewise Smooth Flow algo-



(c) (d)

Figure 3: (a) Frame 8 of the *Yosemite* sequence. (b) Ground-truth and estimated horizontal flow as a function of vertical position, at the sky-ground boundary. (c) Angular error in Brox et al's 2D results. Dark means no error. (d) Angular error after two over-fine iterations.

rithm (Amiaz and Kiryati, 2006).

When running Brox et al's algorithm[1], all images were blurred using a Gaussian kernel before the optical flow estimation was performed. The width of the kernel was dependent on the the number of over-fine iterations performed. For the original algorithm $\sigma = 0.8$ was used, $\sigma = 1.4$ was used for one over-fine step, and $\sigma = 2.6$ and $\sigma = 5.0$ were used for two and three over-fine steps respectively. All other parameters used were taken from Brox et al. (2004) ($\alpha = 80$, $\gamma = 100$, $\eta = 0.95$). The number of internal iterations was 5, the number of SOR iterations 7, and the number of external iterations was chosen according to the size of the image. The implementation of Anandan's algorithm was taken from ftp://ftp.csd.uwo.ca/pub/vision/ANANDAN/. We used a three level image pyramid for the

[1] We used our implementation of Brox et al's algorithm, because the original implementation of their algorithm was not available to us. All results presented in this section are those of our implementation, and differ only slightly from those published by Brox et al. (2004).

original method, and four levels for the over-fine version. The window size for the original method was 3, and 5 for the over-fine version. The increased window size corresponds to a wider blurring kernel. Bi-cubic interpolation was used to create the over-fine images.

Figure 3 shows the spatial distribution of angular error for the *Yosemite with Clouds* sequence. Notice the reduced over-smoothing near the sky-ground interface. Table 1 shows that on the *Yosemite with Clouds* sequence, and with all algorithms tested, significant reduction of average angular error was achieved. The result for the over-fine Piecewise Smooth Flow algorithm (Amiaz and Kiryati, 2006) on the *Yosemite with Clouds* sequence is better than any result previously published[2]. Table 2 presents results on the *Yosemite without Clouds* sequence. The result is slightly better than that of Roth and Black (2005) which to the best of our knowledge was the most accurate result published using only two images. Results which are better than any previously published on a sequence are presented in bold in Tables 1-3.

Figure 4 shows the reduction in error around the car in the *Street with Car* sequence. Table 3 demonstrates the significant reduction in average angular error when using the over-fine method. The over-fine version of the Piecewise Smooth Flow algorithm (Amiaz and Kiryati, 2006) is the best result on this sequence.

We finally tested our algorithm on the MPEG *Flower Garden* sequence, which is a real sequence of translational camera movement. Figure 5 shows that the flow edge is closer to the tree in the foreground when using the over-fine method compared with our implementation of Brox et al.

[2]Since this paper was originally submitted, better results on the *Yosemite with Clouds* sequence were published in Brox et al. (2006).

Yosemite with Clouds

Technique	AAE	STD
Anandan	13.07°	15.39°
Over-fine (x2)	11.53°	15.75°
Brox et al. (2D)	2.36°	7.94°
Over-fine (x2)	2.11°	7.54°
Over-fine (x4)	1.96°	7.24°
Piecewise Smooth Flow	1.64°	5.81°
Over-fine (x2)	**1.48°**	**5.41°**

Table 1: Demonstration of the accuracy enhancement of the over-fine approach when applied to the Anandan (1989), Brox et al. (2004) and Piecewise Smooth Flow (Amiaz and Kiryati, 2006) algorithms on the *Yosemite with Clouds* sequence. (AAE: Average angular error. STD: Standard deviation of the angular error.)

Yosemite without Clouds

Technique	AAE	STD
Brox et al. (2D)	1.63°	1.55°
Over-fine (x2)	1.49°	1.59°
Over-fine (x4)	**1.44°**	**1.55°**

Table 2: Comparison between the Brox et al. algorithm and the coarse to over-fine enhancement on the *Yosemite without Clouds* sequence.

Figure 4: (a) Frame 10 of the *Street with Car* sequence. (b) Angular error in the Piecewise Smooth Flow algorithm. Dark means no error. (c) Angular error after one over-fine iteration.

Street with Car		
Technique	AAE	STD
Anandan	11.29°	15.39°
Over-fine (x2)	10.28°	19.55°
Brox et al. (2D)	3.30°	10.43°
Over-fine (x2)	2.64°	9.35°
Over-fine (x4)	2.37°	8.81°
Piecewise Smooth Flow	2.29°	8.82°
Over-fine (x2)	**1.86°**	**8.76°**

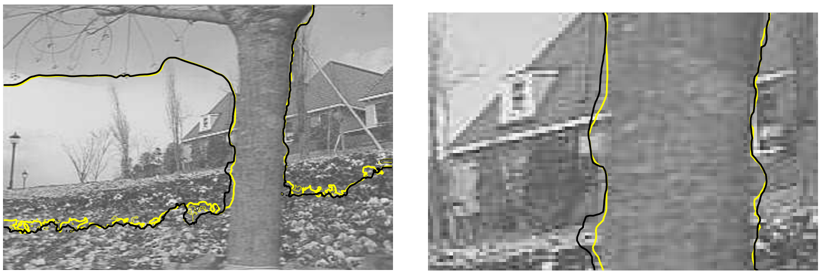
Table 3: Comparison between optical flow algorithms and their enhancement by the coarse to over-fine approach on the *Street with Car* sequence. (AAE: Average angular error. STD: Standard deviation of the angular error.)

3.1 Simulation

One might argue that the over-fine approach is nothing more than parameter tuning, namely reduction of the weight of the smoothness term. To refute this argument we tested the effect of the approach on optical flow estimation of smooth motion. The algorithm of Brox et al. (2004) was the base for the over-fine approach in these simulations. Bi-linear interpolation was used to create the over-fine images.

We prepared sequences of images comprised of sines (Figure 6):

$$I(x,y,t) = \sin(\frac{\pi(x+ut)}{\lambda_1})\sin(\frac{\pi(x+ut)}{\lambda_2})\sin(\frac{\pi y}{\lambda_1})\sin(\frac{\pi y}{\lambda_2}) \ ,$$



(a) (b)

Figure 5: (a) First frame of the *Flower Garden* sequence, the black contour is the result of our implementation of Brox et al., the light contour is from the flow calculated using one step over-fine method. (b) Detail of (a): The tree trunk.

with $\lambda_2 = 48$ (constant), while λ_1 and u are varied.

Figure 7 demonstrates the effect of the over-fine method on estimating optical flow for smooth motions. The improvement is larger at higher frequencies (smaller λ), it is pronounced between 0.2-0.7 pixel movements, and is non-existent at 0.9-1.0 pixel movement. The effect is dominated by the first over-fine step.

4 Discussion

The *coarse to over-fine* approach presented in this paper is an effective way to improve the accuracy of optical flow estimation algorithms. It relies on two complementing advantages: crisper flow discontinuities, and observed higher accuracy for smooth sub-pixel motion. Testing on common test sequences shows that the application of the over-fine approach to multi-scale optical flow estimation algorithms resulted in error reduction by 10%-30%.

We have used simulations to observe the effect of the coarse to over-fine approach on the accuracy of optical flow estimated for smooth motion. Accuracy was observed to improve for motion near half pixel in magnitude, and when the frequencies in the image were high. These happen to be the conditions for recovery of information via super-resolution (Huang

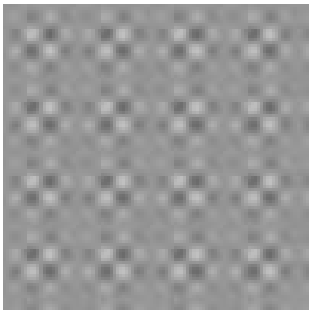
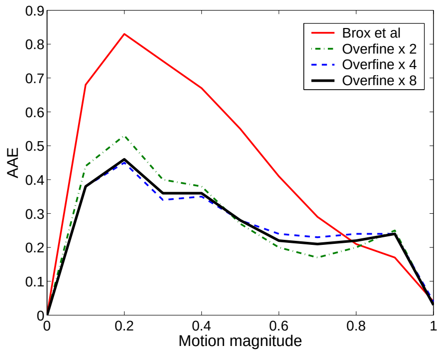
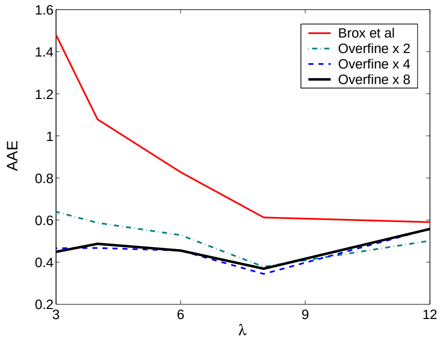


Figure 6: Sample image used for studying the effect of the over-fine approach on optical flow estimation of smooth motion.



(a)



(b)

Figure 7: Demonstration of the coarse to over-fine method on smooth motion. (a) Effect of the over-fine method on the average angular error (AAE) as a function of motion magnitude ($\lambda_2 = 6$). (b) Performance of the over-fine method as a function of λ_2 (motion magnitude 0.2 pixel). As λ_2 decreases the benefit grows.

and Tsay, 1984). Future research should examine the relation between the two. It is also interesting to understand the impact of the method on the data term of the functional in different optical flow estimation algorithms.

The coarse to over-fine approach may be applicable to additional computer vision problems. Optical flow calculation for range images, also known as range flow (Spies et al., 2002), is expected to have the same benefit as optical flow from this approach. Depth from stereo (Beardsley et al., 1996) relies on extracting information from multiple images, and requires piecewise smoothness at object boundaries. This similarity to optical flow makes it likely to benefit from this approach as well.

Recall that the criteria for stopping the coarse to over-fine iteration process (step 6 of the algorithm) rely either on a predefined number of iterations or on the improvement of the objective functional. Future work in this area may incorporate different stopping criteria for different regions in the image. This will enable continuing the coarse to over-fine approach only in areas with higher potential gain, while staying with the current result elsewhere. Determining the iteration limit for each area can be based on either the reduction of the local value of the objective functional, or the local flow properties.

A Proofs

To prove (14) and (15) we begin by showing that a smooth approximation of a sampled sequence, with a single outlier, tends to be farther from the outlier the larger the support.

Consider the minimization problems

$$\vec{z}^1 = \operatorname*{argmin}_{\vec{z}} \left[\sum_{i=1}^{m} (z_i - 1)^2 + \alpha \sum_{i=1}^{m-1} (z_{i+1} - z_i)^2 + \alpha \left(z_m - \theta \right)^2 \right] \tag{18}$$

$$\vec{z}^2 = \operatorname*{argmin}_{\vec{z}} \left[\sum_{i=2}^{m} (z_i - 1)^2 + \alpha \sum_{i=2}^{m-1} (z_{i+1} - z_i)^2 + \alpha \left(z_m - \theta \right)^2 \right] \tag{19}$$

with $\theta < 1$, we show that

$$z_m^1 \geq z_m^2 \ . \tag{20}$$

For the length m minimization problem (18), the minimizing arguments $\{z_i\}$ are determined by differentiating the objective function

$$\sum_{i=1}^{m} (z_i - 1)^2 + \alpha \sum_{i=1}^{m-1} (z_{i+1} - z_i)^2 + \alpha \left(z_m - \theta \right)^2$$

with respect to each z_i and equating to zero. For example, taking the derivative with respect to z_1 we obtain

$$(1+\alpha)z_1 - \alpha z_2 = 1$$

Similarly,

$$\begin{aligned}
&-\alpha z_1 + (1+2\alpha)z_2 - \alpha z_3 = 1 \\
&\qquad\vdots \\
&-\alpha z_{m-2} + (1+2\alpha)z_{m-1} - \alpha z_m = 1 \\
&-\alpha z_{m-1} + (1+\alpha)z_m = \theta \ .
\end{aligned}$$

In matrix form, the resulting system of linear equations is expressed as $A\vec{z} = \vec{y}$ with $\vec{y} = (1, \ldots, 1, \theta)$. Denoting A_m the m-sized matrix, direct calculation shows that $(A_m)^{-1}_{m,m} \leq (A_{m-1})^{-1}_{m-1,m-1}$. This is a sufficient condition for the desired result (20), because $\sum_{i=1}^{m}(A_m)^{-1}_{m,i} = 1$ and

$$\begin{aligned}
z^1_m &= \sum_{i=1}^{m-1}(A_m)^{-1}_{m,i} + (A_m)^{-1}_{m,m}\theta \qquad (21)\\
&= (1-(A_m)^{-1}_{m,m}) + (A_m)^{-1}_{m,m}\theta = 1-(1-\theta)(A_m)^{-1}_{m,m} \\
z^2_m &= 1-(1-\theta)(A_{m-1})^{-1}_{m-1,m-1} \qquad (22)
\end{aligned}$$

Equation (15) has the same form as (18) (with $\theta = 0$, and $m = 2n-1$). Using the relationship (20) recursively we can deduce that $z^*_{of,2n-1} \geq z^*_{of',2n-1}$ where

$$\vec{z}^*_{of'} = \underset{\vec{z}}{\operatorname{argmin}} \left[\sum_{i=n}^{2n-1}(z_i-1)^2 + \alpha\sum_{i=n}^{2n-2}(z_{i+1}-z_i)^2 + \alpha\left(z_{2n-1}\right)^2\right] \ . \qquad (23)$$

But it is obvious that $z^*_{of',2n-1} > z^*_n$ because it is exactly the same smooth approximation problem, only with θ larger for $\vec{z}^*_{of}$ than for $\vec{z}^*$. The same line of reasoning can be applied for (16) and (17).

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References

Amiaz, T., Kiryati, N., 2006. Piecewise-smooth dense optical flow via level sets. International Journal of Computer Vision 68 (2), 111–124.

Anandan, P., January 1989. A computational framework and an algorithm for the measurement of visual motion. International Journal of Computer Vision 2 (3), 283–310.

Bar, L., Sochen, N., Kiryati, N., 2006. Image deblurring in the presence of impulsive noise. International Journal of Computer Vision 70 (3), 279–298.

Barron, J. L., Fleet, D. J., Beauchemin, S. S., February 1994. Performance of optical flow techniques. International Journal of Computer Vision 12 (1), 43–77.

Beardsley, P. A., Torr, P. H. S., Zisserman, A., 1996. 3D model acquisition from extended image sequences. In: 4th European Conference on Computer Vision (ECCV96). Springer-Verlag, pp. Vol II: 683–695.

Black, M. J., Anandan, P., 1996. The robust estimation of multiple motions: parametric and piecewise-smooth flow fields. Computer Vision and Image Understanding 63 (1), 75–104.

Brox, T., Bruhn, A., Papenberg, N., Weickert, J., 2004. High accuracy optical flow estimation based on a theory for warping. In: 8th European Conference on Computer Vision (ECCV04). Vol. IV. pp. 25–36.

Brox, T., Bruhn, A., Weickert, J., 2006. Variational motion segmentation with level sets. In: 9th European Conference on Computer Vision (ECCV06). Vol. I. pp. 471–483.

Bruhn, A., Weickert, J., Schnörr, C., 2005. Lucas/Kanade meets Horn/Schunck: Combining local and global optic flow methods. International Journal of Computer Vision 61 (3), 211–231.

Galvin, B., McCane, B., Novins, K., Mason, D., Mills, S., September 1998. Recovering motion fields: An evaluation of eight optical flow algorithms. In: 9th British Machine Vision Conference (BMVC98). Vol. 1. pp. 195–204.

Horn, B. K. P., 1986. Robot Vision. MIT Press.

Horn, B. K. P., Schunck, B. G., August 1981. Determining optical flow. Artificial Intelligence 17 (1-3), 185–203.

Huang, T., Tsay, R., 1984. Multiple frame image restoration and registration. Advances in Computer Vision and Image Processing 1, 317–339.

Lucas, B. D., Kanade, T., 1981. An iterative image registration technique with an application to stereo vision. In: DARPA Image Understanding Workshop. pp. 121–130.

Mémin, E., Pérez, P., February 2002. Hierarchical estimation and segmentation of dense motion fields. International Journal of Computer Vision 46, 129–155.

Press, W., Teukolsky, S., Vetterling, W., Flannery, B., 1992. Numerical Recipes in C, 2nd Edition. Cambridge University Press, Cambridge, UK.

Roth, S., Black, M. J., 2005. On the spatial statistics of optical flow. In: 10th International Conference on Computer Vision (ICCV05). Vol. 1. pp. 42–49.

Spies, H., Jähne, B., Barron, J. L., 2002. Range flow estimation. Computer Vision and Image Understanding 85, 209–231.